How big is BCI fragment of BCK logic

KATARZYNA GRYGIEL, PAWEł M. IDZIAK and MAREK ZAIONC
Department of Theoretical Computer Science, Faculty of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland.
E-mail: grygiel@tcs.uj.edu.pl; idziak@tcs.uj.edu.pl; zaionc@tcs.uj.edu.pl

Abstract

We investigate quantitative properties of BCI and BCK logics. The first part of the article compares the number of formulas provable in BCI versus BCK logics. We consider formulas built on implication and a fixed set of k variables. We investigate the proportion between the number of such formulas of a given length n provable in BCI logic against the number of formulas of length n provable in richer BCK logic. We examine an asymptotic behaviour of this fraction when length n of formulas tends to infinity. This limit gives a probability measure that randomly chosen BCK formula is also provable in BCI . We prove that this probability tends to zero as the number of variables tends to infinity. The second part of the article is devoted to the number of lambda terms representing proofs of BCI and BCK logics. We build a proportion between number of such proofs of the same length n and we investigate asymptotic behaviour of this proportion when length of proofs tends to infinity. We demonstrate that with probability 0 a randomly chosen BCK proof is also a proof of a BCI formula.

Keywords: BCK and BCI logics, asymptotic probability in logic, analytic combinatorics.

1 Introduction

The results presented in this article are a part of research in which the likelihood of truth is estimated for various propositional logics with a limited number of variables. Probabilistic methods appear to be very powerful in combinatorics and computer science. From a point of view of these methods we investigate a typical object chosen from some set. For formulas in the fixed propositional language, we investigate the proportion between the number of valid formulas of a given length n against the number of all formulas of length n. Our interest lies in finding the limit of that fraction when n tends to infinity. If the limit exists, then it is represented by a real number which we may call *the density* of the investigated logic. In general, we are also interested in finding the *'density'* of some other classes of formulas. Good presentation and overview of asymptotic methods for random Boolean expressions can be found in the paper [10] of Gardy. For the purely implicational logic of one variable (and at the same time simple type systems), the exact value of the density of true formulas was computed by Moczurad, Tyszkiewicz and Zaionc in [21]. The classical logic of one variable and the two connectives of implication and negation was studied in Zaionc [28]; over the same language, the exact proportion between intuitionistic and classical logics was determined by Kostrzycka and Zaionc in [15]. Asymptotic identity between classical and intuitionistic logic of implication has been proved in Fournier, Gardy, Genitrini and Zaionc in [7]. Some variants involving expressions with other logical connectives have also been considered. Genitrini and Kozik have studied the influence of adding the connectors $\vee$ and $\wedge$ to implication in [12], while Matecki in [20] considered the case of the single equivalence connector. For two connectives again, the *and/or* case has already received much attention — see Lefmann and Savický [19], Chauvin, Flajolet, Gardy and Gittenberger [2], Gardy and Woods [9], and Kozik [17]. Let us also mention the survey [10] of Gardy on the probability

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distributions on Boolean functions induced by random Boolean expressions; this survey deals with the whole set of Boolean functions on some finite number of variables.

As already stated, almost all classical tautologies are valid in the intuitionistic logic. At this point a natural problem arises about the borderline of weakening the intutionistic logic so that the density of true formulas in the obtained logic remains one among classical tautologies. This problem can be rephrased as the question about the expression power of implicational logics. In the article we prove that the BCK logic is still 'dense' in the classical one, whereas rejecting the axiom K causes that tautologies of the obtained BCI logic are already negligible among all classical ones.

2 BCK and BCI logics

The logics BCK and BCI are ones of several pure implication calculi. Its name comes from the connection with the combinators B , C , K and I (see [3]). From the perspective of type theory, BCK and BCI can be viewed as the set of types of a certain restricted family of lambda terms, via the Curry–Howard isomorphism. Formally, logics BCK and BCI can be defined, each one separately, as Hilbert systems by three axiom schemes and detachment rule. Namely BCK is based on B , C and K whereas BCI is based on B , C and I where:

(B) $(\varphi \Rightarrow \psi) \Rightarrow ((\chi \Rightarrow \varphi) \Rightarrow (\chi \Rightarrow \psi))$ (prefixing)
(C) $(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow (\psi \Rightarrow (\varphi \Rightarrow \chi))$ (commutation)
(K) $\varphi \Rightarrow (\psi \Rightarrow \varphi)$
(I) $\varphi \Rightarrow \varphi$ (identity)

In BCK we are able to prove I therefore the logic BCI is a subset of the BCK . Let us observe that implicational formulas may be seen as rooted binary trees.

DEFINITION 1
By a formula tree we mean the rooted binary tree in which nodes are labeled by $\Rightarrow$ and have two successors left and right while leaves of the tree are labeled by variables.

DEFINITION 2
With every implicational formula φ we associate the formula tree $G(\varphi)$ in the following way:

- If x is a variable, then $G(x)$ is a single node labelled with x.
- Tree $G(\varphi \Rightarrow \psi)$ is the tree with the new root labelled with $\Rightarrow$ and two subtrees: left $G(\varphi)$ and right $G(\psi)$.

3 λ-calculus as a proof system

Lambda calculus is a standard mechanism for proof system representation for various propositional calculi. By the Curry–Howard isomorphism there is a one-to-one correspondence between provable formulas in intuitionistic implicational logic and types of closed lambda calculus terms. Moreover, proofs of formulas correspond to typable terms. We start with presenting some fundamental concepts of the λ-calculus, as well as with some new definitions used in this article.

DEFINITION 3
Let V be a countable set of variables. The set Λ of λ-terms is defined by the following grammar:

1. every variable is a lambda term,

2. if t and s are lambda terms then ts is a lambda term,
3. if t is a lambda terms and x is a variable then $\lambda x.t$ is a lambda term.

As usual, λ-terms are considered modulo the α-equivalence, i.e. two terms which differ only by the names of bounded variables are considered equal. Observe that λ-terms can be seen as rooted unary–binary trees.

DEFINITION 4
By a lambda tree we mean the following rooted graph with two kinds of edges: undirected and directed. One distinguished node is called the root of the graph. The graph induced by undirected edges is a rooted tree with the distinguished node being the root of it. There are two kinds of internal nodes labelled by @ and by λ. Nodes labelled by @ have two successors left and right. Nodes labelled with λ have only one successor. Leaves of the tree are either labelled by variables or are connected by directed edge with the one of λ nodes placed on the path from it to the root.

DEFINITION 5
With every lambda term t we associate the lambda tree $G(t)$ in the following way:

- If x is a variable then $G(x)$ is a single node labelled with x.
- Lambda tree $G(PQ)$ is a lambda tree with the new root labelled with @ and connected by two new undirected edges with roots of two lambda subtrees left $G(P)$ and right $G(Q)$.
- Tree $G(\lambda x.P)$ is obtained from $G(P)$ in four steps:
    - Add new root node labelled with λ.
    - Connect new root by undirected edge with the root of $G(P)$.
    - Connect all leaves of $G(P)$ labelled with x by directed edges with the new root.
    - Remove all labels x from $G(P)$.

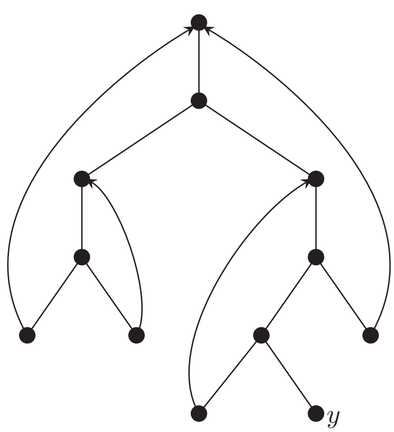

FIGURE 1. The lambda tree representing the term $\lambda z.(\lambda u.zu)((\lambda u.uy)z)$.

OBSERVATION 6
If T is a lambda tree, then $T = G(M)$ for some lambda term M. Terms M and N are α-equivalent iff $G(M) = G(N)$. Free variables of term M are the same as variables labelling leafs of $G(M)$.

We often use (without giving the precise definition) the classical terminology about trees (e.g. path, root, leaf, etc.). A path from the root to a leaf is called a branch.

3.1 BCI and BCK classes of lambda terms

The Curry–Howard isomorphism for proof representation in entire intuitionistic implicational logic can be restricted to weaker logics. Therefore, we can look at the axioms (B), (C), (I) and (K) of BCI and BCK logics as at types à la Curry in the typed lambda calculus. Lambda terms for which these types are principal (the most general ones) are respectively ([13]):

$$\begin{aligned} \mathbf{B} &\equiv \lambda xyz.x(yz), \\ \mathbf{C} &\equiv \lambda xyz.xzy, \\ \mathbf{I} &\equiv \lambda x.x, \\ \mathbf{K} &\equiv \lambda xy.x. \end{aligned}$$

THEOREM 7
The combinator I is provable in BCK .

PROOF. For example the lambda term $(\mathbf{CK})\mathbf{K} = \lambda z.z$ forms proof for the combinator I in logic BCK . ■

THEOREM 8
BCI is a proper subset of BCK

PROOF. Inclusion follows from Theorem 7. Combinator K is not provable in logic BCI (see [1], for example). ■

We are going to isolate the special set of BCK provable formulas called simple tautologies that forms a simple and large fragment of the set of all BCK provable formulas. As we will see afterwards the class of simple tautologies is so big that it can play a role of good approximation of the whole set of BCK tautologies. Therefore, quantitative investigations about behaviour of the whole set can be nicely approximated by this fragment. See [29] for discussion about quantitative aspects of simple tautologies.

DEFINITION 9
A simple simple tautology is an implicational formula of the form $\tau_1 \Rightarrow (\ldots \Rightarrow (\tau_p \Rightarrow \alpha)\ldots)$ such that $p > 0$, α is a variable and there is at least one component τ_i identical to α.

THEOREM 10
Every simple tautology is BCK provable.

PROOF. By $(\mathbf{CK})\mathbf{K}$ we can prove $\alpha \Rightarrow \alpha$. Using several times axiom (K) we can add any number of premisses and prove $\tau_1 \Rightarrow (\ldots \Rightarrow (\tau_p \Rightarrow (\alpha \Rightarrow \alpha))\ldots)$. Using axiom (C) we are able to permute premisses to get $\tau_1 \Rightarrow (\ldots \Rightarrow (\tau_p \Rightarrow \alpha)\ldots)$. ■

DEFINITION 11
The smallest class of lambda calculus terms containing **B**, **C** and **I** (resp. **B**, **C** and **K**) and closed under application and β-reduction is called the class of BCI (resp. BCK) lambda terms.

THEOREM 12
(1) A BCI lambda term is a closed lambda term P such that

- (i) for each subterm $\lambda x.M$ of P, x occurs free in M exactly once,
- (ii) each free variable of P has just one free occurrence in P.

(2) A BCK lambda term is a closed lambda term P such that

(i) for each subterm $\lambda x.M$ of P, x occurs free in M at most once,
(ii) each free variable of P has just one free occurrence in P.

PROOF. Proof can be found in Roger Hindley's book [13]. ■

By Theorem 12 we immediately get that the BCI is a proper subclass of BCK .

4 Classes of formulas

DEFINITION 13
The language F^k over k propositional variables $\{a_1,\ldots,a_k\}$ is defined inductively as:

$$a_i \in F^k \quad \text{for all } i \leq k,$$
$$\phi \Rightarrow \psi \in F^k \quad \text{if } \phi \in F^k \text{ and } \psi \in F^k.$$

We can now define the usual notation for a formula. Let $T \in \mathsf{F}^k$ be a formula. Hence it is of the form $A_1 \Rightarrow (A_2 \Rightarrow (\ldots \Rightarrow (A_p \Rightarrow r(T)))\ldots)$; we shall write it

$$T = A_1,\ldots,A_p \Rightarrow r(T).$$

The formulas A_i are called the *premisses* of T and the rightmost propositional variable $r(T)$ of the formula is called the *goal* of T. For formula T which is itself a propositional variable obviously $p=0$ and $r(T)=T$. To prove quantitative results about BCK and BCI logics, we need to define several other classes of formulas, all of them being special kinds of either tautologies or non-tautologies.

DEFINITION 14
We define the following subsets of F^k:

- The set of all *classical tautologies,* CL^k is the set of formulas that are *true* under any $\{0,1\}$ valuation.
- The set of all *intuitionistic tautologies,* INT^k is the set of formulas for which there are closed lambda terms (constructive proofs) of type identical with the formula.
- The set of all *Peirce formulas,* $PEIRCE^k$ is the set of classical tautologies that are not intuitionistic ones.
- The set BCK^k is the set of formulas for which there are closed lambda BCK terms of type identical with the formula.
- The set BCI^k is the set of formulas for which there are closed lambda BCI terms of type identical with the formula.
- The set of *simple tautologies,* G^k is the set of expressions that can be written as

$$T = A_1,\ldots,A_p \Rightarrow r(T),$$

  where at least one of A_i's is the variable $r(T)$.
- The set of even formulas $EVEN^k$, is the set of formulas in which each variable occurs even number of times.
- The set of simple non-tautologies SN^k, is the set of formulas of the form

$$T = A_1,\ldots,A_p \Rightarrow r(T),$$

where $r(A_i) \neq r(T)$ for all i.

- The set LN^k is the set of less simple non-tautologies, defined as the set of formulas of the form

$$T = B_1, \ldots, B_{i-1}, C, B_i, \ldots, B_p \Rightarrow r(T),$$

such that

$$C = C_1, C_2, \ldots, C_q \Rightarrow r(C),$$

where $r(C) = r(T)$, $q \geqslant 1$, and

$$C_1 = D_1, D_2, \ldots, D_r \Rightarrow r(D),$$

where $r(D) \neq r(T)$, $r \geqslant 0$, and the following holds: for all j, $r(B_j) \notin \{r(T), r(D)\}$ and $r(D_j) \notin \{r(T), r(D)\}$.

The obvious relations between classes above are the following.

LEMMA 15
$\mathsf{SN}^k \cup \mathsf{LN}^k \subseteq \mathsf{F}^k \setminus \mathsf{CL}^k$.

PROOF. Suppose $T = A_1, \ldots, A_p \Rightarrow r(T)$ is in SN^k. Then evaluate, all of assumptions A_i by 1 and goal $r(T)$ by 0 we get that $T \notin \mathsf{CL}$. Now let $T \in \mathsf{LN}^k$ and T is in the form described by the definition 14. The shape of T allows us to evaluate $r(T)$ and $r(D)$ by 0 and all the $r(B_j)$ and $r(D_j)$ by 1 to see that $T \notin \mathsf{CL}$. ■

LEMMA 16
$\mathsf{SN}^k \cap \mathsf{LN}^k = \emptyset$.

PROOF. Simply by observing the syntactic structure of both sets. ■

LEMMA 17
$\mathsf{G}^k \subseteq \mathsf{BCK}^k \subseteq \mathrm{INT}^k \subsetneq \mathrm{CL}^k \subsetneq \mathsf{F}^k \setminus (\mathrm{SN}^k \cup \mathrm{LN}^k)$.

PROOF. The first inclusion can be found in Theorem 10. The rest is trivial. ■

LEMMA 18
$\mathsf{BCI}^k \subseteq \mathsf{EVEN}^k \cap \mathsf{BCK}^k$.

PROOF. $\mathsf{BCI}^k \subsetneq \mathsf{EVEN}^k$ has been proved recently by Tomasz Kowalski in the paper [16] Theorem 6.1. $\mathsf{BCI}^k \subsetneq \mathsf{BCK}^k$ is a classical fact that can be found in [1]. See also Theorem 8. ■

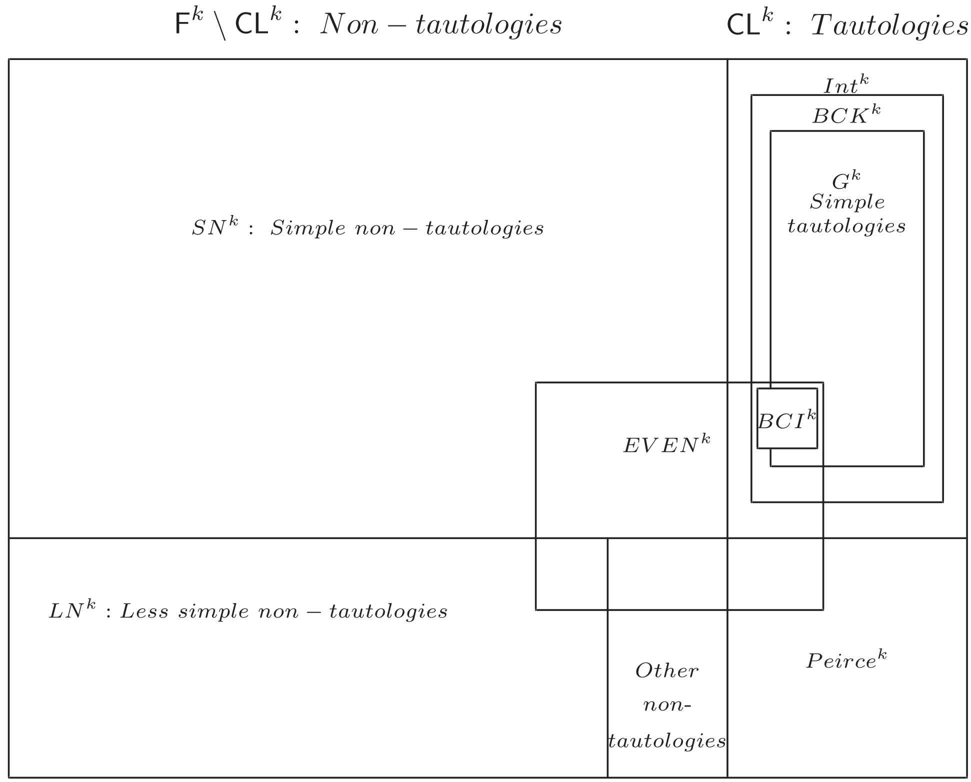

FIGURE 2. Inclusions summarized in Lemmas 15 till 18.

5 Densities of sets of formulas

First we establish the way in which the size of formula trees are measured.

DEFINITION 19
By $\|\phi\|$ we mean the size of formula ϕ which we define as total number of leaves in the formula tree $G(\phi)$. This is in fact the total number of occurrences of propositional variables in the formula. Formally,

$$\|a_i\| = 1 \text{ and } \|\phi \Rightarrow \psi\| = \|\phi\| + \|\psi\|.$$

DEFINITION 20
We associate the density $\mu(\mathbf{X})$ with a subset $\mathbf{X} \subseteq \mathsf{F}^k$ of formulas as:

$$\mu(\mathbf{X}) = \lim_{n\to\infty} \frac{\#\{t \in \mathbf{X} : \|t\| = n\}}{\#\{t \in \mathsf{F}^k : \|t\| = n\}} \tag{1}$$

if the limit exists.

The number $\mu(\mathbf{X})$ if it exists is an asymptotic probability of finding a formula from the class $\mathbf{X}$ among all formulas from F^k or it can be interpreted as the asymptotic density of the set $\mathbf{X}$ in the set F^k. It can be immediately seen that the density μ is finitely additive so if $\mathbf{X}$ and $\mathbf{Y}$ are disjoint classes of formulas such that $\mu(\mathbf{X})$ and $\mu(\mathbf{Y})$ exist then $\mu(\mathbf{X} \cup \mathbf{Y})$ also exists and $\mu(\mathbf{X} \cup \mathbf{Y}) = \mu(\mathbf{X}) + \mu(\mathbf{Y})$.

It is straightforward to observe that for any finite set $\mathbf{X}$ the density $\mu(\mathbf{X})$ exists and is 0. Dually for co-finite sets $\mathbf{X}$ the density $\mu(\mathbf{X})=1$. The density μ is not countably additive so in general the formula

$$\mu\left(\bigcup_{i=0}^{\infty}\mathbf{X}_i\right)=\sum_{i=0}^{\infty}\mu(\mathbf{X}_i) \tag{2}$$

does not hold for all pairwise disjoint classes of sets $\{\mathbf{X}_i\}_{i\in\mathbb{N}}$. A good counterexample for the Equation (2) is to take as $\mathbf{X}_i$ the singelton of i-th formula from our language under any natural order of formulas. On the left-hand side of Equation (2) we get $\mu(\mathsf{F}^k)$, which is 1 but on right-hand side $\mu(\mathbf{X}_i)=0$ for all $i\in\mathbb{N}$ and so the sum is 0. Finally, we define:

$$\mu^-(\mathbf{X}) = \liminf_{n\to\infty}\frac{\#\{t\in\mathbf{X}:\|t\|=n\}}{\#\{t\in\mathsf{F}^k:\|t\|=n\}}$$
$$\mu^+(\mathbf{X}) = \limsup_{n\to\infty}\frac{\#\{t\in\mathbf{X}:\|t\|=n\}}{\#\{t\in\mathsf{F}^k:\|t\|=n\}}.$$

These two numbers are well defined for any set of formulas $\mathbf{X}$, even when the limiting ratio $\mu(\mathbf{X})$ is not known to exist.

5.1 *Enumerating formulas*

In this section, we present some properties of numbers characterizing the amount of formulas in different classes defined in our language. Many results and methods could be rephrased purely in terms of binary trees with given properties. Obviously an implicational formula from F^k of size n can be seen as a binary tree with n leaves and k labels per leaf (see Definitions 1 and 2). We will analyse several classes of formulas (trees).

DEFINITION 21
By F_n^k we mean the total number of formulas from F^k of size n so:

$$\text{F}_n^k=\#\{\phi\in\mathsf{F}^k:\|\phi\|=n\}. \tag{3}$$

LEMMA 22
The number $\text{F}_n^k=k^n\text{C}_n$ where C_n is $(n-1)$-th Catalan number.

PROOF. We may use combinatorial observation. A formula from F^k of size n can be interpreted as full binary tree of n leaves with k label per leaf. Therefore for $n=0$ and $n=1$ it is obvious. Any formula of size $n>1$ is the implication (tree) between some pair of formulas (trees) of sizes i and $n-i$, respectively. Therefore, the total number of such pairs is $\sum_{i=1}^{n-1}F_i^kF_{n-i}^k$. Therefore by simple induction we can immediately see that $F_n^k=k^nC_n$. For more elaborate treatment of Catalan numbers see Wilf [26, pp. 43–44]. We mention only the following well-known non-recursive formula for $C_n=\frac{1}{n}\binom{2n-2}{n-1}$. ■

LEMMA 23
The number G_n^k of simple tautologies is given by the recursion

$$\mathrm{G}_1^k = 0, \ \mathrm{G}_2^k = k, \tag{4}$$

$$\mathrm{G}_n^k = \mathrm{F}_{n-1}^k - \mathrm{G}_{n-1}^k + \sum_{i=2}^{n-1} \mathrm{F}_{n-i}^k \mathrm{G}_i^k. \tag{5}$$

PROOF. For the whole discussion about simple tautologies see [29]. In particular for a proof look at Lemma 15 of [29]. ■

LEMMA 24
The number EVEN_n^k is given by:

$$\frac{\mathrm{C}_n}{2^k} \sum_{j=0}^{k} \binom{k}{j} (k-2j)^n.$$

PROOF. The proof is based on the observation obtained in the paper of Franssens ([8] p. 30, formula 7.18). In this paper it is obtained an explicit formula $e_n^k = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} (k-2j)^n$ for the number of closed walks, based at a vertex, of length n along the edges of k-dimensional cube (see also [23]). This number in fact appears at Maclaurin series of $\cosh^k(t)$, for all k. Note that for odd n this gives 0. As we can see the number e_n^k obviously enumerates the set of all sequences of length n of variables $\{a_1, \ldots, a_k\}$ in which every variable a_i occurs even number of times. Multiplying this by the number C_n of all binary trees with n leaves we obtain the explicit formula for $EVEN_n^k$. For some special k, sequences e_n^k are mentioned in Sloane's catalogue [22]. For instance, $e_{2n}^2 = 2*4^{p-2}$ is described in Sloane as A009117. The sequence $e_{2n}^3 = (3^n+3)/4$ is present in Sloane's catalogue as A054879. Finally e_{2n}^4 is Sloane's A092812. ■

5.2 *Generating functions*

In this article, we investigate the proportion between the number of formulas of the size n that are tautologies in various logics against the number of all formulas of size n for propositional formulas of the language F^k. Our interest lies in finding limit of that fraction when n tends to infinity. For this purpose combinatorics has developed an extremely powerful tool, in the form of generating series and generating functions. A nice exposition of the method can be found in Wilf [26], as well as in in Flajolet, Sedgewick [6]. As the reader may now expect, while working with formulas we will be often concerned with complex analysis, analytic functions and their singularities.

Let $A = (A_0, A_1, A_2, \ldots)$ be a sequence of real numbers. The *ordinary generating series* for A is the formal power series $\sum_{n=0}^{\infty} A_n z^n$. And, of course, formal power series are in one-to-one correspondence to sequences. However, considering z as a complex variable, this series, as known from the theory of analytic functions, converges uniformly to a function $f_A(z)$ in some open disc $\{z \in \mathbb{C} : |z| < R\}$ of maximal diameter, and $R \geq 0$ is called its radius of convergence. So with the sequence A we can associate a complex function $f_A(z)$, called the *ordinary generating function* for A, defined in a neighbourhood of 0. This correspondence is one-to-one again (unless $R = 0$), since, as it is well known from the theory of analytic functions, the expansion of a complex function $f(z)$, analytic in a neighbourhood of z_0, into a power series $\sum_{n=0}^{\infty} A_n (z-z_0)^n$ is unique.

Many questions concerning the asymptotic behaviour of A can be efficiently resolved by analysing the behaviour of its generating function f_A at the complex circle $|z| = R$. This is the approach we take to determine the asymptotic fraction of tautologies and many other classes of formulas among all formulas of a given size.

The main tool used to obtain limits of the fraction of two sequences that are described by generating functions will be the following result, due to Szegö [24] [Thm. 8.4], see as well Wilf [26] [Thm. 5.3.2 p. 181]. We can see versions of Szegö lemma in action in papers [28], [29] concerning asymptotic probabilities in logic. The second powerful tool we need is so called Drmota–Lalley–Woods theorem that has been developed independently by Drmota in [5], Lalley in [18] and Woods in [27] to study problems involving the enumeration of families of plane trees or context-free languages and finding their asymptotic behaviour as solutions of positive algebraic systems. The best presentation of Drmota–Lalley–Woods theorem can be found in Flajolet and Sedgewick in [6] [pp. 446–451]. Excellent overview of the metod is due to Daniele Gardy in [10] [Chapter 4, pp 15–16].

Theorems and lemmas in the next Section 5.3 are proved using Szegö lemma. The result mentioned in Theorems 28 and 31 that the limiting ratio $\mu(\mathsf{CL}^k)$ of classical tautologies with k propositional variables exists, requires the use of Drmota–Lalley–Woods theorem.

5.3 Densities of classes of formulas

In this section, we wish to summarize results contained in the papers [28] and [7].

LEMMA 25
The asymptotic probability of the fact that a randomly chosen formula is a simple tautology is:

$$\mu(\mathsf{G}^k) = \lim_{n\to\infty} \frac{\mathrm{G}_n^k}{\mathrm{F}_n^k} = \frac{4k+1}{(2k+1)^2}.$$

PROOF. The first proof of this fact can be found in [21]. A simpler one is in Theorem 30 of [29] page 252. ■

LEMMA 26
The density of simple non-tautologies exists and is equal to

$$\mu(\mathsf{SN}^k) = \lim_{n\to\infty} \frac{\mathrm{SN}_n^k}{\mathrm{F}_n^k} = \frac{k(k-1)}{(k+1)^2}.$$

For large k, this density is $1 - 3/k + \Theta(1/k^2)$.

PROOF. This result was already given in the paper [21, p. 586]. The alternative proof can be found in [7] at Proposition 7. ■

LEMMA 27
The density of less simple non-tautologies is equal to

$$\mu(\mathsf{LN}^k) = \lim_{n\to\infty} \frac{\mathrm{LN}_n^k}{\mathrm{F}_n^k} = \frac{2k(k-1)^2}{(k+2)^4}.$$

For large k it is equal to $2/k + \Theta(1/k^2)$.

PROOF. The long and complicated proof of this fact can be found in chapter 4.3 of [7]. ■

THEOREM 28
Asymptotically (for a large number k of Boolean variables), all classical tautologies are simple and it follows that all intuitionistic tautologies are classical i.e.

$$\lim_{k\to\infty} \frac{\mu(G^k)}{\mu(CL^k)} = 1,$$
$$\lim_{k\to\infty} \frac{\mu^-(INT^k)}{\mu(CL^k)} = 1.$$

PROOF. We know that for any k, the limiting ratio $\mu(Cl^k)$ of classical tautologies with k propositional variables exists. This result is obtained by standard techniques in analysis of algorithms; we skip the details and refer the interested reader to Flajolet and Sedgewick [6] or to Gardy [10]. From the fact presented in Lemma 17

$$\mathsf{G}^k \subseteq \mathsf{INT}^k \subseteq \mathsf{CL}^k \subseteq \mathsf{F}^k \setminus (\mathsf{SN}^k \cup \mathsf{LN}^k)$$

it follows

$$\mu(G^k) \le \mu^-(\mathsf{INT}^k) \le \mu(\mathsf{CL}^k) \le 1 - \mu(\mathsf{SN}^k) - \mu(\mathsf{LN}^k).$$

Now our result follows since lower and upper bounds $\mu(\mathsf{G}^k)$ and $1-\mu(\mathsf{SN}^k)-\mu(\mathsf{LN}^k)$ are equal to $1/k + \Theta(1/k^2)$. ■

LEMMA 29
The density $\mu(PEIRCE^k)$ of Peirce formulas (if it exists) is equal to $\frac{1}{2k^2}$.

$$\liminf_{n\to\infty} \frac{\mathtt{PEIRCE}_n^k}{F_n^k} = \limsup_{n\to\infty} \frac{\mathtt{PEIRCE}_n^k}{F_n^k} = \frac{1}{2k^2}.$$

PROOF. Proof of this fact appears in the paper [11]. ■

THEOREM 30
For the set $EVEN^k$ of formulas we have:

$$\mu^-(EVEN^k) = \liminf_{n\to\infty} \frac{\mathrm{EVEN}_n^k}{F_n^k} = 0$$
$$\mu^+(EVEN^k) = \limsup_{n\to\infty} \frac{\mathrm{EVEN}_n^k}{F_n^k} = \frac{1}{2^{k-1}}.$$

PROOF. The first equality is trivial since $EVEN_n^k = \frac{C_n}{2^k}\sum_{j=0}^{k}\binom{k}{j}(k-2j)^n$ is zero by Lemma 24 for all odd numbers n. Now suppose that $n=2m$.

$$\begin{aligned}
\limsup_{2m\to\infty}\frac{EVEN_{2m}^k}{F_{2m}^k} &= \lim_{2m\to\infty}\frac{C_{2m}}{2^k}\sum_{j=0}^{k}\binom{k}{j}\frac{(k-2j)^{2m}}{F_{2m}^k}\\
&= \frac{1}{2^k}\sum_{j=0}^{k}\lim_{2m\to\infty}\binom{k}{j}\frac{C_{2m}(k-2j)^{2m}}{k^{2m}C_{2m}}\\
&= \frac{1}{2^k}\lim_{2m\to\infty}\binom{k}{0}\frac{C_{2m}k^{2m}}{k^{2m}C_{2m}}+\frac{1}{2^k}\lim_{2m\to\infty}\binom{k}{k}\frac{C_{2m}(-k)^{2m}}{k^{2m}C_{2m}}\\
&= \frac{1}{2^k}+\frac{1}{2^k}=\frac{1}{2^{k-1}},
\end{aligned}$$

where the second and the third line in the above display are equal as

$$\lim_{2m\to\infty}\binom{k}{j}\frac{C_{2m}(k-2j)^{2m}}{k^{2m}C_{2m}}=0$$

for every $0<j<k$. ■

The picture below summarizes all theorems involving densities that are needed for proving our next two results on BCK and BCI logics namely Theorems 31 and 32.

THEOREM 31
Almost every classical tautology is BCK provable i.e.

$$\lim_{k\to\infty}\frac{\mu^-(BCK^k)}{\mu(\mathsf{CL}^k)}=1.$$

PROOF. From the fact that $\mathsf{G}^k\subseteq\mathsf{BCK}^k\subseteq\mathsf{INT}^k\subseteq\mathsf{CL}^k$, we have

$$\mu(G^k)=\lim_{k\to\infty}\frac{\mathrm{G}_n^k}{\mathrm{F}_n^k}\leqslant\liminf_{n\to\infty}\frac{BCK_n^k}{\mathrm{F}_n^k}\leqslant\limsup_{n\to\infty}\frac{INT_n^k}{\mathrm{F}_n^k}\leqslant\lim_{k\to\infty}\frac{CL_n^k}{\mathrm{F}_n^k}=\mu(CL^k).$$

The result follows from the fact that both $\mu(\mathsf{G}^k)$ and $\mu(\mathsf{CL}^k)$ are equal to $1/k+\Theta(1/k^2)$. ■

THEOREM 32
Almost every BCK provable formula is not BCI provable, i.e.,

$$\lim_{k\to\infty}\frac{\mu^+(BCI^k)}{\mu^-(BCK^k)}=0.$$

PROOF. It follows from two facts. Lemma 18 shows $\mathsf{BCI}^k\subseteq\mathsf{EVEN}^k$ while from Lemma 17 based on Theorem 10 we get $\mathsf{G}^k\subset\mathsf{BCK}^k$. The rest is based on the calculations from Theorem 30 and Lemma 25.

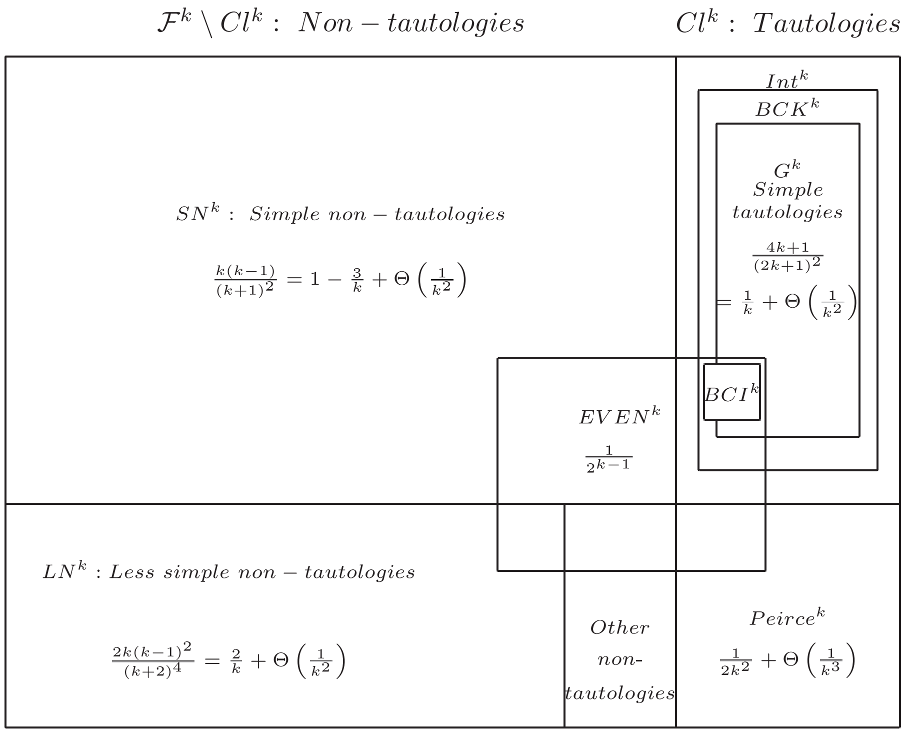

FIGURE 3. Densities of the sets of formulas.

Therefore:

$$\mu^+(\mathsf{BCI}^k) \le \mu^+(\mathsf{EVEN}^k) \le \frac{1}{2^{k-1}},$$

while

$$\mu^-(\mathsf{BCK}^k) \ge \mu(\mathsf{G}^k) = \frac{4k+1}{(2k+1)^2}.$$

Finally we have

$$\lim_{k\to\infty} \frac{\mu^+(\mathsf{BCI}^k)}{\mu^-(\mathsf{BCK}^k)} \le \lim_{k\to\infty} \frac{(2k+1)^2}{2^{k-1}(4k+1)} = 0.$$

■

Interpretation of result: weakening rule $\varphi \Rightarrow (\psi \Rightarrow \varphi)$ is much stronger tool to generate formulas then identity rule $\varphi \Rightarrow \varphi$.

Open problems: we do not know if there exist densities $\mu(\mathsf{BCI}^k)$ or $\mu(\mathsf{BCK}^k)$ of two investigated logics.

6 Counting proofs in BCI and BCK logics

In this section, we will focus on two special classes BCI and BCK of lambda terms. On the basis of their special structure, we will show how to enumerate BCI and BCK terms of a given size. As the main result the density of BCI terms among BCK terms is computed.

DEFINITION 33
The size of a lambda term is defined in the following way:

$$\|x\| = 1$$
$$\|\lambda x.M\| = 1 + \|M\|$$
$$\|MN\| = 1 + \|M\| + \|N\|.$$

As we can see $\|t\|$ is the number of all nodes of lambda tree $G(t)$.

DEFINITION 34
Let n be an integer. We denote by Λ_n the set of all closed lambda terms up to α conversion of size n. Obviously the set Λ_n is finite. We denote its cardinality by L_n.

As far as we know, no asymptotic analysis of the sequence L_n has been done. Moreover, typical combinatorial techniques do not seem to apply easily for this task. For the first time the problem of enumerating lambda terms was considered in [25]. In general, counting lambda terms of a given size turns out to be a non-trivial and challenging task. The wide discussion on this problem and some results concerning properties of random terms can be found in [4].

6.1 Enumerating BCI terms

DEFINITION 35
By a_n we denote the number of BCI terms of size n.

Since in BCI terms each lambda binds exactly one variable, in a lambda tree for such a term the number of leaves is equal to the number of unary nodes. In every unary–binary tree the number of leaves is greater by one than the number of binary nodes. Thus, the number of BCI terms is positive only if the size is equal to $3k+2$ (k binary nodes, $k+1$ unary nodes and $k+1$ leaves) for $k \in \mathbb{N}$. Therefore, $a_n = 0$ for $n \neq 2 \bmod 3$.

DEFINITION 36
By a_n^* we denote the number of BCI terms up to α conversion with n binary nodes.

Obviously, $a_n^* = a_{3n+2}$.

LEMMA 37
The sequence (a_n^*) satisfies the recurrence:

$$a_0^* = 1, \qquad a_1^* = 5,$$
$$a_n^* = 6na_{n-1}^* + \sum_{i=1}^{n-2} a_i^* a_{n-i-1}^* \quad , \text{ for } n \geq 2.$$

PROOF. There is only one BCI term of size 2 (no binary nodes): $\lambda x.x$. Moreover there are five terms of size 5 (one binary node): $\lambda xy.xy$, $\lambda xy.yx$, $(\lambda x.x)(\lambda x.x)$, $\lambda x.(\lambda y.y)x$ and $\lambda x.x(\lambda y.y)$. Thus, $a_0^* = 1$ and $a_1^* = 5$.

Let P be a BCI term with $n \geq 2$ binary nodes. Such a term is either in the form of application or in the form of abstraction. Both cases are depicted in Figure 4.

In the first case P is an application of two BCI terms, $P \equiv MN$, where M has i binary nodes and N has $n-i-1$ binary nodes ($i = 0, \ldots, n-1$). It gives us $\sum_{i=0}^{n-1} a_i^* a_{n-i-1}^*$ possibilities.

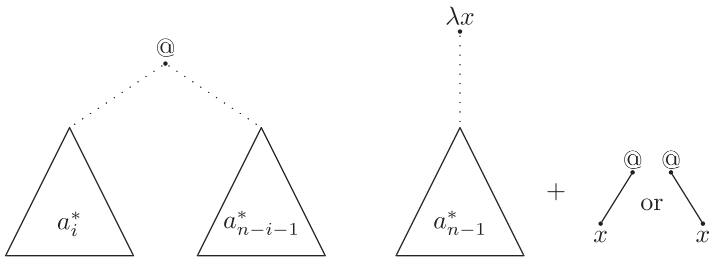

FIGURE 4. Two ways of obtaining a BCI term with $n \geq 2$ binary nodes.

In the second case P is in the form of abstraction, $P \equiv \lambda x.M$, and x occurs free in M exactly once. In the tree corresponding to M, the parent of the leaf labelled with x must be a binary node. Thus, we can look at the tree corresponding to M as at the lambda tree for some BCI term Q with $n-1$ binary nodes with an additional leaf labelled with x. This leaf can be inserted into the tree in two manners, either on the left or on the right. Moreover, this can be done in $3n-1$ ways which is the number of all branches in the tree for Q. Thus, there are $2(3n-1)a_{n-1}^*$ possibilities of such insertions.

Summing up we get the equation of the lemma. ■

Denoting by $A(x)$ the generating function for the sequence (a_n^*) and after basic calculations, we get

$$6x^2 \frac{\partial A(x)}{\partial x} + xA^2(x) + (4x-1)A(x) + 1 = 0, \qquad A(0) = 1.$$

This is a non-linear Riccati differential equation and as such it has a solution that is a non-elementary function.

The sequence (a_n^*) and the function $A(x)$ were studied in [14]. On the basis of that paper we get the asymptotics

$$a_{3n+2} = a_n^* \sim \frac{1}{2\pi} 6^n (n-1)!.$$

First values of (a_n) are the following:

$$0,0,1,0,0,5,0,0,60,0,0,1105,0,0,27120,0,0,828250,0,0,30220800,\ldots$$

This sequence can be found in the On-Line Encyclopedia of Integer Sequences ([22]) under the number A062980.

6.2 *Enumerating BCK terms*

DEFINITION 38
Let us denote by b_n the number of BCK lambda terms of size n.

LEMMA 39
The sequence (b_n) satisfies the following recursive equation:

$$b_0 = b_1 = 0, \qquad b_2 = 1, \qquad b_3 = 2, \qquad b_4 = 3,$$
$$b_n = b_{n-1} + 2\sum_{i=0}^{n-3} i b_i + \sum_{i=0}^{n-1} b_i b_{n-i-1} + 1 \quad \text{for } n \geq 5.$$

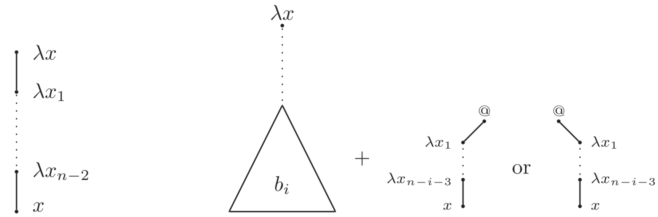

FIGURE 5. Second case of the construction of a BCK term of size $n \geq 5$.

PROOF. There are no terms of size 0 and 1, there is only one BCK term of size 2: $\lambda x.x$, two terms of size 3: $\lambda xy.x$ and $\lambda xy.y$, and three terms of size 4: $\lambda xyz.x$, $\lambda xyz.y$ and $\lambda xyz.z$. Thus $b_0 = b_1 = 0$, $b_2 = 1$, $b_3 = 2$ and $b_4 = 3$.

Let P be a BCK term of size $n \geq 5$. Such a term is either in the form of application or in the form of abstraction where the first lambda binds one variable or in the form of abstraction where the first lambda does not bind any variables.

In the first case P is in the form of application, $P \equiv MN$, where M and N are BCK terms of size, respectively, i and $n-i-1$ ($i = 0, \ldots, n-1$). It gives us $\sum_{i=0}^{n-1} b_i b_{n-i-1}$ possibilities.

In the second case P is in the form of abstraction, $P \equiv \lambda x.M$ and x occurs free in M exactly once. There are two subcases here: either $M \equiv \lambda x_1 \ldots x_{n-2}.x$ or M is a term built of a BCK term Q of size $i = 2, \ldots, n-3$ with an additional term $\lambda x_1 \ldots x_{n-i-3}.x$ inserted on one of its branches or on the branch joining λx with Q. Since in Q there are $i-1$ branches and the additional term can be inserted either on the left or on the right, this case gives us $1 + 2\sum_{i=2}^{n-3} i b_i$ possibilities. Both subcases are presented in Figure 5.

In the third case P is in the form of abstraction, $P \equiv \lambda x.M$ and x does not occur free in M. The number of such terms of size n is equal to the number of all BCK terms of size $n-1$. Thus, it gives us b_{n-1} possibilities.

Summing up we get the equations of the lemma. ■

Denoting by $B(x)$ the generating function for the sequence (b_n) and after basic calculations, we get

$$2x^4 \frac{\partial B(x)}{\partial x} + (x - x^2)B^2(x) - (1-x)^2 B(x) + x^2 = 0, \qquad B(0) = 0.$$

Again we obtained a non-linear Riccati differential equation. Unfortunately, this time we do not know the solution.

First values of (b_n) are the following:

$$0, 0, 1, 2, 3, 9, 30, 81, 225, 702, 2187, 6561, 19602, 59049, 177633, 532170, 1594323, \ldots$$

Also this sequence can be found in [22] under the number A073950.

6.3 *Density of BCI in BCK*

Our main goal is to compute the density of BCI in BCK. Since there are no BCI terms of size $n \not\equiv 2 \mod 3$ (which is not true in the case of BCK terms), if the density exists, then it is 0.

Let us observe that each BCK term can be obtained from a BCI term with some additional (possibly none) lambdas. This observation allows us to obtain a formula for b_n depending on the sequence (a_n).

LEMMA 40
The number of possible ways of choosing k out of the n elements with repetition is equal to $\binom{n+k-1}{n-1}$.

LEMMA 41
For $k \in \mathbb{N}$ the following formula holds

$$b_{3k+2} = \sum_{i=0}^{k} \binom{3k}{3i} a_{3i+2}.$$

PROOF. Each BCK term of size $3k+2$ can be obtained from a BCI term of size $3i+2$ ($i=0,\ldots,k$) by inserting $3k+2-3i-2=3k-3i$ additional lambdas. There are $3i+1$ branches in the tree for a BCI term of size $3i+2$, thus the number of insertions corresponds to the number of $3k-3i$-element combinations with repetition of a $3i+1$-element set and thus, by Lemma 40, it is equal to $\binom{3k}{3k-3i} = \binom{3k}{3i}$. ■

THEOREM 42
The density of BCI terms among BCK terms equals 0.

PROOF. By the asymptotics of (a_{3k+2}) and by Lemma 41, we get

$$\frac{a_{3k+2}}{b_{3k+2}} \leq \frac{a_{3k+2}}{a_{3k+2} + \binom{3k}{3} a_{3k-1}} \stackrel{k\to\infty}{\longrightarrow} 0.$$

■

7 Summary

Our quantitative analysis of logic shows the role and strength of particular axioms in propositional logics. It follows that the most powerful one is the weakening. Indeed, from this point of view we have shown two things. The first one says that almost surely a formula is a classical tautology because it is simple (in the sense of Definition 9). All of these simple tautologies are present in BCK logic, and therefore BCK is dense in classical logic. The second one is that the density of the BCI logic in the BCK one is zero with respect both to tautologies and to their proofs.

Intuitively, this phenomenon depends on the acceptance of the axiom K, or equivalently the weakening rule in Gentzen style formalization. This weakening allows to add as many ‘unnecessary’ premisses as possible to stay within the prescribed size of the formula. Moreover these ‘unnecessary’ premisses can be renamed in all possible ways. In particular, they need not be relevant to the conclusion. This leads to a significant increase of the number of provable formulas. Our analysis also explains the role of the combinator K when looking at the striking difference between BCI and BCK lambda terms. On the other hand, rejection of the axiom K (like in the BCI logic) causes a strong constraint for a formula to be provable. This relevance constraint was postulated in so called relevant logics like R, RM or E. Our quantitative results may support this philosophical position. However, our motivation for these studies is to explore the strength of particular axioms and rules in propositional logic and combinators in proof theory.

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