Boolean functions over implication: probability and complexity

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Outline

- Tautologies
- Density and complexity
Tautologies
  - Formulas over implication, the functions they compute, tautologies
  - Main result: most tautologies are \textit{simple}
  - Corollary: Classical and intuitionistic logic are asymptotically identical

Density and complexity
  - Expansion and pruning
  - Conjecture: Relationship between density and complexity of a function
Context
Representation of Boolean functions

- Variables: $x_1, x_2, x_3, \ldots$
- A set of connectors: \{and, or\} for example (\{\land, \lor\})
Representation of Boolean functions

- Variables: $x_1, x_2, x_3, \ldots$
- A set of connectors: \{and, or\} for example (\{\land, \lor\})
- A formula is a tree:

\[
\begin{array}{c}
\land \\
\lor \\
\lor \\
\land \\
x_1 \\
x_2 \\
x_3 \\
x_3
\end{array}
\]

*Example:* this formula (tree) computes

\[
(x_1 \lor (x_2 \land x_3)) \land x_3.
\]
Many trees for the same function

- Previous tree computes

\[(x_1 \lor (x_2 \land x_3)) \land x_3 = (x_1 \lor x_2) \land x_3.\]
Many trees for the same function

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- This new tree computes the same function:

```
                  ∧
                 /   /
                ∧   x_3
               /     /
             ∨     x_3
            /       /
          x_1     x_2
```
Many trees for the same function

- Previous tree computes

\[(x_1 \lor (x_2 \land x_3)) \land x_3 = (x_1 \lor x_2) \land x_3.\]

- This new tree computes the same function:

- The *size* of a tree is the number of its leaves.
Properties of the system

- Density of a function $f$:
  the proportion of trees which computes $f$
  (if it exists).
Properties of the system

- **Density** of a function $f$:
  the proportion of trees which computes $f$
  (if it exists).

- **Complexity** of a function $f$:
  the size of one of the smallest trees computing $f$.
Properties of the system

- **Density** of a function $f$:
  the proportion of trees which computes $f$
  (if it exists).

- **Complexity** of a function $f$:
  the size of one of the smallest trees computing $f$.

- Does exist a relationship between these characteristics?
Other works: And/Or trees

- Random And/Or trees
  - Two binary connectors: $\land$, $\lor$
  - $k$ variables
  - each leaf is labelled by a literal:
    \[ \{ x_1, \ldots, x_k, \overline{x}_1, \ldots, \overline{x}_k \} \]
  - Uniform distribution over all trees of size $n$
Other works: And/Or trees

- Random And/Or trees
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  - $k$ variables
  - each leaf is labelled by a literal:
    $\{x_1, \ldots, x_k, \overline{x}_1, \ldots, \overline{x}_k\}$
  - Uniform distribution over all trees of size $n$
- Existence of limit distribution $\pi_k$ on Boolean functions
  - $\pi_k$ distribution on Boolean functions $B_k$
  - $\pi_k(f) = \lim_{n \to \infty} P_{T \in T_n}[T \sim f]$
  - Existence obtained by pruning techniques
- [Lefman, Savický. *Some typical properties of large And/Or boolean formulas*, 1997]
And/Or trees: probability, complexity

- $L(f)$: size of one of the smallest tree computing $f$
- First bounds:

$$\frac{1}{4} \left(\frac{1}{8k}\right)^{L(f)} \leq \pi_k(f) \leq e^{-cL(f)/k^3} \left(1 + O\left(\frac{1}{k}\right)\right)$$

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  - [Lefmann, Savický. Some typical properties of large And/Or boolean formulas, 1997]

- Improved upper bounds:
  - $\pi_k(f) \leq e^{-cL(f)/k^2} (1 + O(1/k))$
  - [Chauvin, Flajolet, Gardy, Gittenberger. And/Or trees revisited, 2005]
Tautologies on And/Or trees

- Probability of tautologies
  - Improvement on the result from the previous slide
    - \( \frac{1}{16k} \leq \pi_k(true) \)
  - [Gardy, Woods. 2005]
Tautologies on And/Or trees

- Probability of tautologies
  - Improvement on the result from the previous slide
  - \( \frac{1}{16k} \leq \pi_k(\text{true}) \)
  - [Gardy, Woods. 2005]

- Conjecture: most tautologies are simple
  - Simple: \( \lor \)-path from the root to both \( \ell \) and \( \bar{\ell} \) for some literal \( \ell \)
  - [Woods, 2005]
Trees over implication
Trees over implication

- $k$ positive literals: $x_1, x_2, \ldots, x_k$. 
$
\begin{itemize}
\item k$ positive literals: $x_1, x_2, \ldots, x_k$.
\item A single (binary) connector: implication ($\rightarrow$).
\end{itemize}

Recall: $x_2 \rightarrow x_1$ computes $\overline{x_2} \lor x_1$. 
Trees over implication

- \( k \) positive literals: \( x_1, x_2, \ldots, x_k \).
- A single (binary) connector: implication (\( \rightarrow \)).
  *Recall*: \( x_2 \rightarrow x_1 \) computes \( \overline{x_2} \lor x_1 \).
- Example:

  ![Tree Diagram](image)

  this tree computes \( (x_1 \rightarrow (x_2 \rightarrow x_1)) \rightarrow x_3 \).
Canonical form of a tree

- Decomposition along the right branch
  - $A_1, A_2, \ldots, A_p$ are trees called \textit{premises}
  - $\alpha$ is a variable called the \textit{goal}
Canonical form of a tree

- Decomposition along the right branch
  - $A_1, A_2, \ldots, A_p$ are trees called premises
  - $\alpha$ is a variable called the goal

$$A_1 \rightarrow (A_2 \rightarrow (\ldots \rightarrow (A_p \rightarrow \alpha)) \ldots)$$

Computes $\overline{A_1} \lor \overline{A_2} \lor \ldots \lor \overline{A_p} \lor \alpha$. 
Functions computed in this system

- Notations:
  - $\mathcal{F}_k$ is the set of trees over *implication* and $k$ variables.
  - $F \in \mathcal{F}_k$ computes a Boolean function
    $\{0, 1\}^k \rightarrow \{0, 1\}$.
  - $\mathcal{B}_k$ is the set of Boolean functions with $k$ variables.
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- Implication trees compute exactly all functions of the form
  $\varphi(x_1, \ldots, x_k) \lor x_i$.

- One of the Post classes (called $S_0$).
Aims

- Trees computing the constant function *true* are called *tautologies*.
Aims

- Trees computing the constant function \textit{true} are called \textit{tautologies}.

How are composed most tautologies?

Proportion of tautologies?
Density

- For a subset $A \subset \mathcal{F}_k$ we define

$$\mu(A) = \lim_{n \to \infty} \frac{|\{t \in A, |t| = n\}|}{|\{t \in \mathcal{F}_k, |t| = n\}|},$$

if it exists.

$\mu(A)$ is the asymptotic density of $A$ in $\mathcal{F}_k$. 
Density

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$$

if it exists.

$\mu(\mathcal{A})$ is the asymptotic density of $\mathcal{A}$ in $\mathcal{F}_k$.

- Using the theorem of Drmota, Lalley and Wood, we conclude that tautologies have a density.
Simple tautologies

- Simple tautologies are all trees of the form

\[
\vdash \\
A_1 \rightarrow \\
A_2 \rightarrow \\
\alpha \rightarrow \\
A_p \rightarrow \\
\alpha
\]
Simple tautologies

- Simple tautologies are all trees of the form

- Conjecture: Most tautologies are simple.
The structure of tautologies

- Simple tautologies: $G_k$
- All (classical) tautologies: $Cl_k$
- $G_k \subset Cl_k$
The structure of tautologies

- Simple tautologies: $G_k$
- All (classical) tautologies: $Cl_k$
- $G_k \subset Cl_k$
- Main result: most tautologies are simple

$$\lim_{k \to \infty} \frac{\mu(G_k)}{\mu(Cl_k)} = 1$$
Generating functions

- Objects $\mathcal{X}$
- Each object $x \in \mathcal{X}$ has a size $|x| \in \mathbb{N}$
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- Each object $x \in \mathcal{X}$ has a size $|x| \in \mathbb{N}$
- $\mathcal{A} \subset \mathcal{X}$
- (Ordinary) generating function $g$ of $\mathcal{A}$:

$$g(z) = \sum_{i=0}^{\infty} c_n z^n$$

where $c_n = |\{a \in \mathcal{A}, |a| = n\}|$
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- Equivalently:

$$g(z) = \sum_{a \in \mathcal{A}} z^{|a|}$$
Generating functions of all trees

- $k$ Boolean variables $x_1, \ldots, x_k$
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- Number of trees of size $n$:

$$C_{n-1} \cdot k^n,$$

where $C_{n-1}$ is the $(n-1)$th Catalan number.
Generating functions of all trees

- $k$: Boolean variables $x_1, \ldots, x_k$
- Number of trees of size $n$:
  \[ C_{n-1} \cdot k^n, \]
  where $C_{n-1}$ is the $(n - 1)$th Catalan number.
- Generating function $e$ of all trees satisfies:
  \[ e(z) = k z + e^2(z). \]

We obtain:
\[ e(z) = \frac{1 - \sqrt{1 - 4kz}}{2} \]
Example

- Example with $k = 2$:
  - 6 functions $true, x_1, x_2, x_1 \rightarrow x_2, x_2 \rightarrow x_1, x_1 \lor x_2$
  - $G_f$ : generating function of trees computing $f$
Example

- Example with $k = 2$:
  - 6 functions $true$, $x_1$, $x_2$, $x_1 \rightarrow x_2$, $x_2 \rightarrow x_1$, $x_1 \lor x_2$
- $G_f$ : generating function of trees computing $f$
- System satisfied by the GF:

\[
\begin{align*}
\vdots \\
G_{x_1 \rightarrow x_2} &= G_{x_1} G_{x_2} + G_{true} G_{x_1 \rightarrow x_2} \\
&\quad + G_{x_1 \lor x_2} G_{x_2} + G_{x_1} G_{x_1 \rightarrow x_2} \\
\vdots
\end{align*}
\]
Example

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- $G_f$ : generating function of trees computing $f$
- System satisfied by the GF:

\[
\begin{align*}
G_{x_1 \rightarrow x_2} &= G_{x_1} G_{x_2} + G_{true} G_{x_1 \rightarrow x_2} \\
&+ G_{x_1 \lor x_2} G_{x_2} + G_{x_1} G_{x_1 \rightarrow x_2}
\end{align*}
\]

- Polynomial system of the form

\[(G_{true}, \ldots, G_{x_1 \lor x_2}) = \Phi(G_{true}, \ldots, G_{x_1 \lor x_2})\]
Existence of densities

- Polynomial system \((G_1, \ldots, G_p) = \Phi(G_1, \ldots, G_p)\)
- Drmota-Lalley-Woods theorem
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*Unique solution* \((y_1, \ldots, y_q)\), *all GF have the same radius of convergence* \(\rho < \infty\)
Existence of densities

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*Unique solution* \((y_1, \ldots, y_q)\), *all GF have the same radius of convergence* \(\rho < \infty\)

\(G_i \text{ behaves like } \alpha_i - \beta_i \sqrt{1 - z/\rho} \text{ near } \rho\)
Existence of densities

- Polynomial system \((G_1, \ldots, G_p) = \Phi(G_1, \ldots, G_p)\)
- Drmota-Lalley-Woods theorem

Unique solution \((y_1, \ldots, y_q)\), all GF have the same radius of convergence \(\rho < \infty\)

\(G_i\) behaves like \(\alpha_i - \beta_i \sqrt{1 - z/\rho}\) near \(\rho\)

Consequence: densities exist for all functions
- In particular for the constant function \(true\)
Tautologies’ density for small values of $k$

- Numerical solutions of a quadratic system
  - $k=1$: 0.72
  - $k=2$: 0.52
  - $k=3$: 0.40
  - $k=4$: 0.30
Tautologies’ density for small values of $k$

- Numerical solutions of a quadratic system
  - $k=1$: 0.72
  - $k=2$: 0.52
  - $k=3$: 0.40
  - $k=4$: 0.30

- Systems become quickly large ($k=4$: 942 functions)
  - Smaller systems by partitioning functions into classes
Enumerating simple tautologies

- Simple tautologies are all trees of the form:

```
→
A_1 →
A_2 →
α →
A_p α
```
Enumerating simple tautologies

- Simple tautologies are all trees of the form:

- GF of simple tautologies:

\[
\frac{1}{1 - (e(z) - z)} \cdot z \cdot \frac{1}{1 - e(z)} \cdot z \cdot k
\]
Enumerating simple tautologies (2)

- GF of simple tautologies:

\[
g(z) = \frac{kz^2}{(1 - e(z))(1 - (e(z) - z))} = \frac{P(z) - (1 + z)\sqrt{1 - 4kz}}{2(1 + k + z)}
\]

where \( P \) is a polynomial.
Enumerating simple tautologies (2)

- GF of simple tautologies:

\[ g(z) = \frac{kz^2}{(1 - e(z))(1 - (e(z) - z))} = \frac{P(z) - (1 + z)\sqrt{1 - 4kz}}{2(1 + k + z)} \]

where \( P \) is a polynomial

- Density of simple tautologies:

\[ \mu(G_k) = \lim_{n \to \infty} \frac{|G_k^n|}{|F_k^n|} = \frac{4k + 1}{(2k + 1)^2} \sim \frac{1}{k} \]
\[ F_k \setminus \text{Cl}_k : \text{Non-tautologies} \quad \text{Cl}_k : \text{Tautologies} \]

\[ G_k \]

Simple tautologies

\[ \frac{4k+1}{(2k+1)^2} \]

\[ = \frac{1}{k} + O\left(\frac{1}{k^2}\right) \]
Simple non tautologies

- Tree such that \( r(A_i) \neq \alpha \) for all \( i \)
Simple non tautologies

- Tree such that $r(A_i) \neq \alpha$ for all $i$

\[
\begin{array}{c}
\text{\scriptsize \begin{array}{c}
A_1 \rightarrow \\
\vdots \\
A_2 \rightarrow \\
\vdots \\
A_p \rightarrow \\
\alpha
\end{array}}
\end{array}
\]

- It computes the function:

\[
\left( r(A_1) \lor \ldots \lor r(A_p) \lor \alpha \right) \land \ldots
\]
Simple non tautologies

- Tree such that \( r(A_i) \neq \alpha \) for all \( i \)

- It computes the function:

\[
\left( \overline{r(A_1)} \lor \ldots \lor \overline{r(A_p)} \lor \alpha \right) \land \ldots
\]

- Density: \( 1 - 3/k + O(1/k^2) \)
\[
F_k \setminus Cl_k : \text{Non \textendash tautologies}
\]

\[
SN_k : \text{Simple non \textendash tautologies}
\]

\[
\frac{k(k-1)}{(k+1)^2} = 1 - \frac{3}{k} + O\left(\frac{1}{k^2}\right)
\]

\[
Cl_k : \text{Tautologies}
\]

\[
G_k
\]

\[
\text{Simple tautologies}
\]

\[
\frac{4k+1}{(2k+1)^2}
\]

\[
= \frac{1}{k} + O\left(\frac{1}{k^2}\right)
\]
$T$ a tree with goal $\alpha$: 
$T$ a tree with goal $\alpha$:

- $T$ has *no* premise with goal $\alpha$
- $T$ is not a tautology; density $1 - 3/k$
Summary

$T$ a tree with goal $\alpha$:

- $T$ has no premise with goal $\alpha$
  - $T$ is a not a tautology; density $1 - 3/k$

- $T$ has one premise equal to $\alpha$
  - $T$ is a (simple) tautology; density $1/k$
Summary

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- $T$ has one premise equal to $\alpha$
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- $T$ has several premises with goal $\alpha$
  - The density of these trees is $O(\frac{1}{k^2})$
Summary

$T$ a tree with goal $\alpha$:

- $T$ has no premise with goal $\alpha$
  - $T$ is a not a tautology; density $1 - \frac{3}{k}$
- $T$ has one premise equal to $\alpha$
  - $T$ is a (simple) tautology; density $\frac{1}{k}$
- $T$ has several premises with goal $\alpha$
  - The density of these trees is $O\left(\frac{1}{k^2}\right)$
- $T$ has exactly one premise with goal $\alpha$, not equal to $\alpha$
  - Density $\frac{2}{k}$; How many tautologies?
Less simple non tautologies

\[ A_1 \rightarrow A_2 \rightarrow B \rightarrow A_{p-1} \rightarrow A_p \]

\[ \alpha \neq \gamma \]

\[ X \text{ underlined means } r(X) \notin \{\alpha, \gamma\} \]
Less simple non tautologies

\[ A_1 \rightarrow A_2 \rightarrow B \rightarrow A_{p-1} \rightarrow A_p \rightarrow \alpha \]
\[ C_1 \rightarrow B_2 \rightarrow C_2 \rightarrow B_3 \rightarrow C_4 \rightarrow B_4 \rightarrow B_q \rightarrow \alpha \]

\[ \alpha \neq \gamma \]

\[ X \text{ underlined means } r(X) \notin \{\alpha, \gamma\} \]
Less simple non tautologies (2)

- Computes the function:

\[
\left( \bigvee_i \overline{r(A_i)} \lor \alpha \lor \bigvee_i \overline{r(C_i)} \lor \gamma \right) \land \ldots
\]
Less simple non tautologies (2)

- Computes the function:

\[
\left( \bigvee_{i} r(A_i) \lor \alpha \lor \bigvee_{i} r(C_i) \lor \gamma \right) \land \ldots
\]

- Not a tautology
Less simple non tautologies (2)

- Computes the function:

\[ \left( \bigvee_i r(A_i) \lor \alpha \lor \bigvee_i r(C_i) \lor \gamma \right) \land \ldots \]

- Not a tautology

- Enumeration via the basic constructions

- Density: \( \frac{2}{k} + O(\frac{1}{k^2}) \)
\[ F_k \setminus Cl_k : \text{Non} - \text{tautologies} \]

\[ Cl_k : \text{Tautologies} \]

\[ SN_k : \text{Simple non} - \text{tautologies} \]

\[ \frac{k(k-1)}{(k+1)^2} = 1 - \frac{3}{k} + O\left(\frac{1}{k^2}\right) \]

\[ FN_k : \text{Less simple} \]

\[ \text{non} - \text{tautologies} \]

\[ \frac{2k(k-1)^2}{(k+2)^4} = \frac{2}{k} + O\left(\frac{1}{k^2}\right) \]

\[ G_k \]

\[ \text{Simple} \]

\[ \text{tautologies} \]

\[ \frac{4k+1}{(2k+1)^2} = \frac{1}{k} + O\left(\frac{1}{k^2}\right) \]

\[ = \frac{63}{4k^2} + O\left(\frac{1}{k^3}\right) \]
Result

- (Classical) tautologies: $Cl_k \subset F_k \setminus (SN_k \cup FN_k)$
Result

- (Classical) tautologies: $\text{Cl}_k \subset F_k \setminus (SN_k \cup FN_k)$
- $\mu(\text{Cl}_k) \leq 1 - (1 - \frac{3}{k} + \frac{2}{k} + O(\frac{1}{k^2})) = \frac{1}{k} + O(\frac{1}{k^2})$
Result

- (Classical) tautologies: $\text{Cl}_k \subset F_k \setminus (SN_k \cup FN_k)$
- $\mu(\text{Cl}_k) \leq 1 - \left(1 - \frac{3}{k} + \frac{2}{k} + O\left(\frac{1}{k^2}\right)\right) = \frac{1}{k} + O\left(\frac{1}{k^2}\right)$
- Most tautologies are simple

$$\lim_{k \to \infty} \frac{\mu(G_k)}{\mu(\text{Cl}_k)} = 1$$
Intuitionistic logic

- Deduction rules for propositional calculus:
  - Axioms
    \[ G, A \vdash A \]
  - Introduction of \( \rightarrow \)
    \[ G, A \vdash B \]
    \[ G, A \rightarrow B \]
  - Elimination of \( \rightarrow \) (Modus Ponens)
    \[ G \vdash A \quad G \vdash A \rightarrow B \]
    \[ G \vdash B \]
Intuitionistic vs. classical tautologies

- Formulas proven in this system: intuitionistic tautologies $I_k$
- $Cl_k$: classical tautologies
Intuitionistic vs. classical tautologies

- Formulas proven in this system: intuitionistic tautologies $I_k$
- $Cl_k$: classical tautologies
- $I_k \subset Cl_k$
Intuitionistic vs. classical tautologies

- Formulas proven in this system: intuitionistic tautologies $I_k$
- $Cl_k$: classical tautologies
- $I_k \subset Cl_k$
- $I_k \subsetneq Cl_k$
- Pierce formulas: $\text{Pierce}_k := Cl_k \setminus I_k$
- Example: $((A \rightarrow B) \rightarrow A) \rightarrow A$
Intuitionistic vs. classical tautologies

- Formulas proven in this system: intuitionistic tautologies $I_k$
- $Cl_k$: classical tautologies
  - $I_k \subset Cl_k$
  - $I_k \not\subseteq Cl_k$
  - Pierce formulas: $Pierce_k := Cl_k \setminus I_k$
  - Example: $((A \rightarrow B) \rightarrow A) \rightarrow A$
- $G_k \subset I_k$
Intuitionistic vs. classical tautologies

- Formulas proven in this system: intuitionistic tautologies \( I_k \)
- \( Cl_k \): classical tautologies
- \( I_k \subset Cl_k \)
- \( I_k \nsubseteq Cl_k \)
  - Pierce formulas: \( Pierce_k := Cl_k \setminus I_k \)
  - Example: \( ((A \rightarrow B) \rightarrow A) \rightarrow A \)
- \( G_k \subset I_k \)
- Corollary: most classical tautologies are intuitionistic

\[
\lim_{k \to \infty} \frac{\mu(I_k)}{\mu(Cl_k)} = 1
\]
Density and complexity
Key idea

- Most tautologies are simple
Key idea

- Most tautologies are simple

- Conjecture: Most trees computing a function are simple
Densities of small functions

- \( f = \alpha \).
- To get a lower bound of the density
  - We build the next family \( \mathcal{F}_\alpha \) of trees:
Densities of small functions

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Densities of small functions

- \( f = \alpha \).
- To get a lower bound of the density
  - We build the next family \( \mathcal{F}_\alpha \) of trees:

\[
\begin{array}{c}
A \quad \alpha
\end{array}
\]

with \( A \):

\[
\begin{array}{c}
A_1 \\
A_2 \\
\vdots \\
A_p \\
\alpha \\
\alpha \\
\beta
\end{array}
\]

\( \alpha \neq \beta \)

- The tree computes \( \alpha \lor \overline{A} = \alpha \lor (\alpha \land \ldots) = \alpha \)
Densities of small functions

- \( f = \alpha \).
- To get a lower bound of the density
  - We build the next family \( F_\alpha \) of trees:
    - with \( A \):
      - or a s. tautology
        \[
        \alpha \neq \beta
        \]
        - The tree computes \( \alpha \lor \overline{A} = \alpha \lor (\alpha \land \ldots) = \alpha \)
        - or the tree computes \( \alpha \lor \overline{A} = \alpha \lor 0 = \alpha \).
Densities of small functions (2)

- **Lower bound:**
  \[ \mu(\mathcal{F}_\alpha) = \frac{1}{2k^2} + O\left(\frac{1}{k^3}\right) \]
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  - $\mu(\mathcal{F}_\alpha) = \frac{1}{2k^2} + O\left(\frac{1}{k^3}\right)$
  - $\mu(f) \geq \mu(\mathcal{F}_\alpha) = \frac{1}{2k^2} + O\left(\frac{1}{k^3}\right)$
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- Upper bound: necessary conditions for a tree to compute \( f \)
  \[ A \text{ a tree.} \quad [A] \text{ the function it computes.} \]
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  - $A$ a tree. $[A]$ the function it computes.
  - $[A] \geq r(A)$. As $f = \alpha$, $r(A) = \alpha$
  - So, if $A$ is reduced to a leaf, then $A = \alpha$, else
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Trees over implication – p.38/53
Densities of small functions(2)

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  - \([A] = \overline{A_1} \lor \cdots \lor \overline{A_p} \lor \alpha \) and \([A] = \alpha \)
  - Every premise \( A_i \) satisfies
  - \([\overline{A_i}] \leq \alpha \) ie \([A_i] \geq \overline{\alpha}\).
Upper bound

Every $A_i$ computes something like:

$$\overline{B_1} \lor \cdots \lor \overline{B_q} \lor r(A_i)$$
Densities of small functions(3)

- Upper bound
  - Every $A_i$ computes something like:
    $$\overline{B_1} \lor \cdots \lor \overline{B_q} \lor r(A_i)$$
  - After decomposition of every $B_j$:
    $$\overline{r(B_1)} \lor \cdots \lor \overline{r(B_q)} \lor r(A_i)) \land \cdots \geq \alpha$$
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  - $\forall i, \exists j \mid r(B_j) \in \{\alpha, r(A_i)\}$
Densities of small functions(3)

- **Upper bound**
  - Every $A_i$ computes something like:
    $$\overline{B_1} \lor \cdots \lor \overline{B_q} \lor r(A_i)$$
  - After decomposition of every $B_j$:
    $$\left( \overline{r(B_1)} \lor \cdots \lor \overline{r(B_q)} \lor r(A_i) \right) \land \cdots \geq \alpha$$
  - $\forall i, \exists j \mid r(B_j) \in \{\alpha, r(A_i)\}$
  - $\ldots$
  - When enough conditions, upper bound:
    $$\mu(f) \leq \frac{1}{2k^2} + O\left(\frac{1}{k^3}\right)$$
Densities of small functions (3)

- Upper bound
  - Every $A_i$ computes something like:
    \[
    \overline{B_1} \lor \cdots \lor \overline{B_q} \lor r(A_i)
    \]
  - After decomposition of every $B_j$:
    \[
    (r(B_1) \lor \cdots \lor r(B_q) \lor r(A_i)) \land \cdots \geq \alpha
    \]
  - $\forall i, \exists j \mid r(B_j) \in \{\alpha, r(A_i)\}$
  - \ldots
  - When enough conditions, upper bound:
    \[
    \mu(f) \leq \frac{1}{2k^2} + O\left(\frac{1}{k^3}\right)
    \]
    \[
    \mu(f = \alpha) = \frac{1}{2k^2} + O\left(\frac{1}{k^3}\right)
    \]
Summary of densities

- \( \mu(f = \alpha) = \frac{1}{2k^2} + O\left(\frac{1}{k^3}\right) \)
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- \( \mu(f = \alpha) = \frac{1}{2k^2} + O\left(\frac{1}{k^3}\right) \)
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Minimal trees:

\[
\begin{align*}
\alpha & \to \\
\alpha & \to \\
\alpha & \to \\
\beta & \to
\end{align*}
\]

\[
\begin{align*}
\alpha & \to \\
\alpha & \to \\
\beta & \to \\
\beta & \to \\
\alpha & \to
\end{align*}
\]

\[
\begin{align*}
\alpha & \to \\
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- Minimal trees:

  \[
  \begin{array}{c}
  \alpha \\
  \alpha \\
  \alpha \rightarrow \\
  \beta \\
  \beta \\
  \beta \\
  \alpha \\
  \beta \\
  \alpha
  \end{array}
  \]

- Conjectures:
  - \( \mu(f) = \frac{\lambda_f}{k^{L(f)+1}} + O(\frac{1}{k^{L(f)+2}}) \)
Summary of densities

- $\mu(f = \alpha) = \frac{1}{2k^2} + O\left(\frac{1}{k^3}\right)$
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- **Minimal trees:**

- **Conjectures:**
  - $\mu(f) = \frac{\lambda_f}{k^{L(f)+1}} + O\left(\frac{1}{k^{L(f)+2}}\right)$
  - Most trees computing $f$ are simple
What does *simple* mean here?

- $f$: a function different from 1
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- $\text{Min}(f)$: the set of the minimal trees of $f$
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- 3 rules to expand the minimal trees
What does *simple* mean here?

- \( f \): a function different from 1
- \( \text{Min}(f) \): the set of the minimal trees of \( f \)
- 3 rules to expand the minimal trees

*We get most trees computing \( f \) with only one expansion*
Expansion rules

- $f \neq 1$ a function
- $A$ a minimal tree of $f$, $|A| = L(f)$
Expansion rules

- $f \neq 1$ a function
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- $\nu$ a node of $A$
Expansion rules

- \( f \neq 1 \) a function
- \( A \) a minimal tree of \( f \), \( |A| = L(f) \)
- \( \nu \) a node of \( A \)

becomes after expansion:

\[
\begin{array}{c}
E \\
/ \\
L \\
/ \\
L
\end{array} \rightarrow
\begin{array}{c}
E \\
/ \\
L \\
/ \\
L
\end{array}
\]

where \( E \) is a valid expansion
Valid expansions

- $E$ is a valid expansion if
  - $E$ is a tautology; new tree computes $f$
Valid expansions

- $E$ is a valid expansion if
  - $E$ is a tautology; new tree computes $f$
  - $E$ has a goal $\alpha$, the new tree computes $f$ and if we change $E$ with any tree with goal $\alpha$ then the new tree computes $f$
Valid expansions

- \( E \) is a valid expansion if
  - \( E \) is a tautology; new tree computes \( f \)
  - \( E \) has a goal \( \alpha \), the new tree computes \( f \) and if we change \( E \) with any tree with goal \( \alpha \) then the new tree computes \( f \)
  - \( E \) has a premise reduce to \( \alpha \), the new tree computes \( f \) and if we change \( E \) with any tree with a premise \( \alpha \) then the new tree computes \( f \)
Valid expansions

- $E$ is a valid expansion if
  - $E$ is a tautology; new tree computes $f$
  - $E$ has a goal $\alpha$, the new tree computes $f$ and if we change $E$ with any tree with goal $\alpha$ then the new tree computes $f$
  - $E$ has a premise reduce to $\alpha$, the new tree computes $f$ and if we change $E$ with any tree with a premise $\alpha$ then the new tree computes $f$

- 3 rules for pruning: inverse rules
Example

- $f = x_1 \rightarrow x_2$, a single minimal tree

\[ \xymatrix{ & \rightarrow \\
\rightarrow & \rightarrow \\
x_1 & x_2 } \]
Example

- \( f = x_1 \rightarrow x_2 \), a single minimal tree

Valid expansions:

- \( E \): tautology
- \( E \): goal \( x_1 \)
- \( E \): premise \( x_2 \)
Example

\[ f = x_1 \rightarrow x_2, \text{ a single minimal tree} \]

Valid expansions:

\[
\begin{align*}
E: & \text{ tautology} \\
E: & \text{ goal } x_1 \\
E: & \text{ premise } x_2
\end{align*}
\]

\[
\begin{align*}
E: & \text{ tautology} \\
E: & \text{ goal } x_2 \\
E: & \text{ premise } x_1
\end{align*}
\]
Example

- \( f = x_1 \to x_2 \), a single minimal tree

Valid expansions:

- \( E: \) tautology
- \( E: \) goal \( x_1 \)
- \( E: \) premise \( x_2 \)
- \( E: \) tautology
- \( E: \) goal \( x_2 \)
- \( E: \) premise \( x_1 \)
- \( E: \) tautology
- \( E: \) goal \( x_1 \)
- \( E: \) premise \( x_2 \)
Example

- \( f = x_1 \rightarrow x_2 \), a single minimal tree

Valid expansions:

- \( E: \text{tautology} \)
- \( E: \text{goal } x_1 \)
- \( E: \text{premise } x_2 \)
- \( E: \text{tautology} \)
- \( E: \text{goal } x_2 \)
- \( E: \text{premise } x_1 \)
- \( E: \text{tautology} \)
- \( E: \text{goal } x_1 \)
- \( E: \text{premise } x_2 \)

\[ \mu(f = x_1 \rightarrow x_2) = \frac{9}{16k^3} + O\left(\frac{1}{k^4}\right) \]
Properties of pruning

- Irreducible trees
Properties of pruning

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- $\{\text{minimal trees}\} \subsetneq \{\text{irreducible trees}\}$
Properties of pruning

- Irreducible trees

\[ \{ \text{minimal trees} \} \subsetneq \{ \text{irreducible trees} \} \]

Example:

\[ f = x_1 \lor (\overline{x}_2 \land \overline{x}_3) \lor (\overline{x}_2 \land x_4) \]
Properties of pruning (2)

- The system is not confluent
Properties of pruning (2)

- The system is not confluent
Aim

- Conjecture:
  Most trees computing $f$ are given by a single expansion in minimal trees.
Aim

- **Conjecture:**
  Most trees computing $f$ are given by a single expansion in minimal trees.

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Aim

- **Conjecture:**
  Most trees computing $f$ are given by a single expansion in minimal trees.

- Several pruning of a tree computing $f$ give an irreducible tree.

- Several expansions of all irreducible trees computing $f$ give all trees computing $f$.

\[ \mu \left( \bigcup_i E^*(\text{Irr}(f)) \right) = \mu(f) \]
Definitions

- $f$ a function $\neq 1$
  - Essential / inessential variables of $f$
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- $P_1(f)$: $A$ irreducible, $|A| = L(f) + i$
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- $P_5(f)$: $A$ i., $|A| = L(f) + e + i$, $e > 1$
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- Conjecture: $\mu(E^1(P_0(f))) \sim \mu(\bigcup_i E^*(P_i(f))) \sim \mu(f)$
Recall

- \( f = x_1 \rightarrow x_2 \), one minimal tree

Valid expansions:

- \( E: \text{tautology} \)
- \( E: \text{goal } x_1 \)
- \( E: \text{premise } x_2 \)

\[
\mu(f = x_1 \rightarrow x_2) = \frac{9}{16k^3} + O\left(\frac{1}{k^4}\right)
\]
What we have to prove now

- $P_0(f)$: minimal trees
- $P_1(f)$: $A$ irreducible, $|A| = L(f) + i$
- $P_2(f)$: $A$ irreducible, $|A| = L(f) + 1$, no inessential v.
- $P_3(f)$: $A$ i., $|A| = L(f) + 1 + i$, all occurrences of i.v. ≠
- $P_4(f)$: $A$ i., $|A| = L(f) + 1 + i$, 1 occ. of i.v. repeated
- $P_5(f)$: $A$ i., $|A| = L(f) + e + i$, $e > 1$

- First result (easy): $\mu\left(E^1(P_0(f))\right) = \frac{\lambda_f}{kL(f)+1} + O\left(\frac{1}{kL(f)+2}\right)$

- Conjecture: $\mu\left(E^1(P_0(f))\right) \sim \mu\left(\bigcup_i E^*(P_i(f))\right) \\ \sim \mu(f)$
Implication and negative literals

- Binary trees over the connector $\rightarrow$
- Leaves are labelled with $x_1, x_2, \ldots x_k, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k$
- Two kinds of simple tautologies:

\[
\begin{align*}
A_1 \rightarrow A_2 \rightarrow A_p & \quad A_1 \rightarrow \ell \rightarrow \overline{\ell} \rightarrow A_p \\
A_1 \rightarrow \ell \rightarrow A_p & \quad A_1 \rightarrow \overline{\ell} \rightarrow \overline{\ell} \rightarrow A_p
\end{align*}
\]
Implication and negative literals (2)

- Enumeration of the different classes
  - simple tautologies
  - simple non tautologies
  - less simple non tautologies
- Classes are union of (not disjoint) simple subclasses
  - Inclusion-Exclusion principle
  - Developed far enough to have the first term
- Result: most tautologies are simple
  - Their density is equal to:

\[
\frac{7}{8k} + O \left( \frac{1}{k^2} \right)
\]
Further work

- Asymptotic development for the density of tautologies?
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- Pierce formulas \( (\text{Cl}_k \setminus I_k) \)
  - Existence of density? Value?
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- Trees over other connectives
  - When does it happen that:
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Further work

- Asymptotic development for the density of tautologies?
- Pierce formulas \((Cl_k \setminus I_k)\)
  - Existence of density? Value?
- Trees over other connectives
  - When does it happen that: “most tautologies are simple”?
- Complexity and probability
  - Relationship between \(\mu(f)\) and \(L(f)\)
  - Conjecture: \(\mu(f) \sim \frac{\alpha f}{kL(f)+1}\)